Darboux integrability for 3D Lotka-Volterra systems

2000 J. Phys. A: Math. Gen. 33 2395

(http://iopscience.iop.org/0305-4470/33/12/307)

J. Phys. A: Math. Gen. **33** (2000) 2395–2406. Printed in the UK PII: S0305-4470(00)05934-5

Darboux integrability for 3D Lotka–Volterra systems

Laurent Cairó† and Jaume Llibre‡

† Département de Mathématiques MAPMO, UMR 6628, Université d'Orléans, BP 6759, 45067 Orléans, Cédex 2, France
‡ Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193, Bellaterra, Barcelona, Spain

E-mail: lcairo@labomath.univ-orleans.fr and jllibre@mat.uab.es

Received 12 July 1999, in final form 21 October 1999

Abstract. We describe the improved Darboux theory of integrability for polynomial ordinary differential equations in three dimensions. Using this theory and computer algebra, we study the existence of first integrals for the three-dimensional Lotka–Volterra systems. Only working up to degree two with the invariant algebraic surfaces and the exponential factors, we find the major part of the known first integrals for such systems, and in addition we find three new classes of integrability. The method used is of general interest and can be applied to any polynomial ordinary differential equations in arbitrary dimension.

1. Introduction

Nonlinear ordinary differential equations appear in many branches of applied mathematics and physics. In dimensions greater than two these systems usually present chaotic motion in the sense that they depend sensitively on the choice of initial conditions, and more specifically the difference between the initial conditions grows exponentially with time. It is important to find conditions for the absence of this chaotic motion by looking for parameter values for which the systems can be partially or completely integrable. For three-dimensional systems the existence of one first integral means that the system is partially integrable, and the existence of two independent first integrals means that the system is completely integrable (because the phase portrait is then completely characterized). If a three-dimensional system is integrable its solutions have good behaviour and it is possible to obtain global information on its long-term evolution. Since the notion of integrability is based on the existence of first integrals the following natural question arises. Given a system of ordinary differential equations depending on parameters, how does one recognize the values of the parameters for which the system has first integrals? Many different methods have been used to study the existence of first integrals. Some of them have been developed for Hamiltonian systems, such as the Ziglin [1, 2] analysis, or the method based on the Noether symmetries [3]. Other methods can be applied to non-Hamiltonian systems: the method of Darboux [4], the method of Lie symmetries [5], the Painlevé analysis [6], the use of Lax pairs [7], the direct method [8], the linear compatibility analysis method [9], the Carlemann embedding procedure [10, 11], the quasimonomial formalism [12], etc.

In 1878 Darboux [4] showed how one could construct the first integrals of planar polynomial ordinary differential equations possessing sufficient invariant algebraic curves. In particular, he proved that if a planar polynomial ordinary differential system of degree m

(see section 2 for a definition) has at least $[m(m+1)/2]+1$ invariant algebraic curves, then it has a first integral, which has an easy expression as a function of the invariant algebraic curves. The version of the Darboux theory of integrability for three-dimensional polynomial vector fields that we summarize in theorem 2 (see section 2), improves Darboux's original exposition because we also take into account the exponential factors introduced by Christopher [13] (see [14] for more details and proofs), and the independent singular points [15]. The proofs given in [14] are for two-dimensional polynomial vector fields but the arguments are the same for any dimension greater than two. The Darboux theory of integrability works for real or complex polynomial ordinary differential equations, but in this paper we only consider real systems and we only study their real first integrals. The Darboux method for finding time-independent first integrals has been used by several authors (see, for instance, [13, 16–18]), and it can also be applied to the search for time-dependent first integrals (see [18–20]). In this work we restrict our interest to finding time-independent first integrals.

We want to show that the Darboux method of integrability is one of the best methods for finding first integrals of polynomial ordinary differential equations. In so doing, we choose the three-dimensional Lotka–Volterra system (without the quadratic self-interacting terms) as a paradigmatic system for the study of the integrability and show that not only can one obtain easily almost all the previous known first integrals for such systems but also find new cases of integrability. This model introduced by Volterra [21] and Lotka [22] appears in ecology where it models a three-species competition, and it has been widely used in applied mathematics and in a large variety of problems in physics: laser physics, plasma physics (where it approximates the Vlasov–Poisson equation), convective instabilities, neural networks, etc (see the references in Almeida *et al* [23]). These authors have examined the integrability of the three-dimensional Lotka–Volterra systems by using the method of Lie symmetries. A more complete study of the integrability of the three-dimensional Lotka–Volterra systems has been made by Grammaticos *et al* [24] using the linear compatibility method, the Painlevé analysis and the Jacobi last-multiplier method. These systems were studied in arbitrary dimension and with the quadratic self-interacting terms by Cairó *et al* [25] and Cairó and Feix [26] using the Carlemann method. The polynomial first integrals of the three-dimensional homogeneous Lotka–Volterra system have been analysed using the Darboux theory of integration by Labrunie [27] and Moulin-Ollagnier [28]. The rational first integrals of degree zero of the three-dimensional homogeneous Lotka–Volterra system has been characterized recently by Moulin-Ollagnier [34].

Intimately associated with the three-dimensional Lotka–Volterra systems are the so-called ABC systems, which correspond to the particular case where the linear terms are absent. Between these systems there is a known simple relation which we recall below. The *three-dimensional Lotka–Volterra systems* considered here are

$$\begin{aligned}\frac{dx}{dt} &= \dot{x} = P(x,y,z) = x(\lambda + Cy + z)\\ \frac{dy}{dt} &= \dot{y} = Q(x,y,z) = y(\mu + x + Az)\\ \frac{dz}{dt} &= \dot{z} = R(x,y,z) = z(\nu + Bx + y)\end{aligned} \tag{1}$$

where we note the absence of the quadratic self-interacting terms. We are concerned here with the existence of (time-independent) first integrals of (1) when the six parameters λ, μ, ν, A, B, C, the three dependendent variables x, y, z, and the independent variable t (the *time*) are real.

If $\lambda = \mu = \nu \neq 0$ then the change of variables $(x, y, z, t) \to (\bar{x}, \bar{y}, \bar{z}, \bar{t})$ given by

$$\bar{x} = x e^{-\lambda t} \qquad \bar{y} = y e^{-\lambda t} \qquad \bar{z} = z e^{-\lambda t} \qquad \bar{t} = \frac{1}{\lambda} e^{\lambda t} \tag{2}$$

transforms (1) into the form

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}\bar{t}} = \bar{x}(C\bar{y} + \bar{z})$$
$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{t}} = \bar{y}(\bar{x} + A\bar{z})$$
$$\frac{\mathrm{d}\bar{z}}{\mathrm{d}\bar{t}} = \bar{z}(B\bar{x} + \bar{y}).$$

This particular class of three-dimensional Lotka–Volterra systems are called the *ABC systems* (see, for instance, Labrunie [27]). Therefore, the dynamics of the three-dimensional Lotka–Volterra systems with $\lambda = \mu = \nu \neq 0$ is equivalent to the dynamics of the same systems with $\lambda = \mu = \nu = 0$, i.e. the ABC systems.

This paper is organized as follows. In section 2 we present the results of the Darboux theory of integrability adapted to the three-dimensional polynomial differential systems. The first integrals of a polynomial ordinary differential system constructed using the Darboux theory are based in the invariant algebraic surfaces and the exponential factors that the system has. Thus, for the three-dimensional Lotka–Volterra systems we study in section 3 their invariant algebraic surfaces $f(x, y, z) = 0$, where f is a polynomial of degree at most two, and in section 4 their exponential factors $\exp(g/h)$ with g and h being polynomials of degree at most two. In section 5 we give the first integrals and the integrable systems (i.e. systems having two independent first integrals) of the three-dimensional Lotka–Volterra systems obtained using the invariant algebraic surfaces and the exponential factors computed in the previous two sections, theorems 6 and 7 summarize our main results. In section 6 we compare our results with the known results. Finally, we give our conclusions in section 7.

2. Darboux integrability theory

Before stating the main results of the Darboux theory for three-dimensional polynomial vector fields we need some definitions.

In this paper a *polynomial vector field* X is a vector field in $\mathbb{R}^3$ of the form

$$X = P(x, y, z)\frac{\partial}{\partial x} + Q(x, y, z)\frac{\partial}{\partial y} + R(x, y, z)\frac{\partial}{\partial z}$$

where P, Q and R are polynomials in the variables x, y and z with real coefficients. In all of this section $m = \max\{\deg P, \deg Q, \deg R\}$ will denote the *degree* of the polynomial vector field X.

Here we say that $H : U \to \mathbb{R}^3$ is a *first integral* of the vector field X if the Lebesgue measure of $\mathbb{R}^3 \setminus U$ is zero and H is a non-constant analytic function which is constant on all solution surfaces $(x(t), y(t), z(t))$ of the vector field X on U; i.e. $H(x(t), y(t), z(t)) =$ constant for all values of t for which the solution $(x(t), y(t), z(t))$ is defined on U. Clearly, H is a first integral of the polynomial vector field X on U if and only if $XH = 0$ on all the points (x, y, z) of U. If H is a first integral of X, then we can reduce the study of the trajectories of X on the invariant sets $H(x, y, z) = h$ when h varies in $\mathbb{R}$. We note that if $h \in \mathbb{R}$ is a regular value of the function H, then $H(x, y, z) = h$ is a surface of $\mathbb{R}^3$, and that by Sard's theorem the regular values are dense in $\mathbb{R}$.

We say that the vector field X is *integrable* if X has two independent first integrals; i.e. if X has two first integrals $H_i : U_i \to \mathbb{R}^3$ for $i = 1, 2$ such that the two vectors

$$\left(\frac{\partial H_1}{\partial x}, \frac{\partial H_1}{\partial y}, \frac{\partial H_1}{\partial z}\right) \qquad \left(\frac{\partial H_2}{\partial x}, \frac{\partial H_2}{\partial y}, \frac{\partial H_2}{\partial z}\right)$$

are independent at all the points $(x, y, z) \in U_1 \cap U_2$ except perhaps into a subset of zero Lebesgue measure. If X is integrable with the two independent first integrals H_1 and H_2, then its trajectories are determined by intersecting the invariant sets $H_1(x, y, z) = h_1$ and $H_2(x, y, z) = h_2$ when h_1 and h_2 vary in $\mathbb{R}$. Hence, the dynamics (i.e. the trajectories) of an integrable system is very well understood.

Let $f \in \mathbb{R}[x, y, z]$, where as usual $\mathbb{R}[x, y, z]$ denotes the ring of the polynomials in the variables x, y and z with real coefficients. The algebraic surface $f = 0$ is called an *invariant algebraic surface* of the polynomial vector field X if for some polynomial $K \in \mathbb{R}[x, y, z]$ we have

$$Xf = P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = Kf. \tag{3}$$

The polynomial K is called the *cofactor* of the invariant algebraic surface $f = 0$. We note that since the polynomial vector field has degree m, then any cofactor has at most degree $m - 1$.

Since on the points of an invariant algebraic surface $f = 0$ the gradient $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)$ is orthogonal to the polynomial vector field $X = (P, Q, R)$ (see (3)), it follows that at every point (x, y, z) of the surface $f = 0$ the vector field X is contained into the tangent plane to the surface $f = 0$ at that point. Hence, the surface $f = 0$ is formed by trajectories of the vector field X. This justifies the name 'invariant algebraic surface' given to the algebraic surface $f = 0$ satisfying (3) for some polynomial K, because it is *invariant* under the flow defined by X.

Let $g, h \in \mathbb{R}[x, y, z]$ be relatively prime polynomials in the ring $\mathbb{R}[x, y, z]$. Then the function $\exp(g/h)$ is called an *exponential factor* of the polynomial vector field X if the equality

$$X\left(\exp\left(\frac{g}{h}\right)\right) = K \exp\left(\frac{g}{h}\right) \tag{4}$$

is satisfied for some polynomial $K \in \mathbb{R}[x, y, z]$ of degree at most $m - 1$. As before we say that K is the *cofactor* of the exponential factor $\exp(g/h)$ (see [13, 14]), where the exponential factors are introduced as a limit of suitable invariant algebraic surfaces.

From the point of view of the integrability of polynomial vector fields the importance of the exponential factors is twofold. On one hand, they verify equation (4), and on the other hand, their cofactors are polynomials of degree at most $m - 1$. These two facts allow them to play the same role as the invariant algebraic surfaces in the integrability of a three-dimensional polynomial vector field X. We note that the exponential factors do not define invariant surfaces of the flow of the vector field X.

The following proposition is due to Christopher [13].

Proposition 1. *If $F = \exp(g/h)$ is an exponential factor for the polynomial vector field X, then $h = 0$ is an invariant algebraic surface, and g satisfies the equation*

$$Xg = gK_h + hK_F$$

where K_h and K_F are the cofactors of h and F, respectively.

Before stating the main results of the Darboux theory we need some definitions. If $S(x, y, z) = \sum_{i+j+k=0}^{m-1} a_{ijk}x^i y^j z^k$ is a polynomial of degree $m - 1$ with $(m + 2)(m + 1)m/6$ coefficients in $\mathbb{R}$, then we write $S \in \mathbb{R}_{m-1}[x, y, z]$. We identify the linear vector space $\mathbb{R}_{m-1}[x, y, z]$ with $\mathbb{R}^{(m+2)(m+1)m/6}$ through the isomorphism

$$S \rightarrow (a_{000}, a_{100}, a_{010}, a_{001}, \ldots, a_{m-1,0,0}, a_{m-2,1,0}, \ldots, a_{0,0,m-1}).$$

We say that r points $(x_k, y_k, z_k) \in \mathbb{R}^3$, $k = 1, \ldots, r$, are *independent* with respect to $\mathbb{R}_{m-1}[x, y, z]$ if the intersection of the r hyperplanes

$$\sum_{u+v+w=0}^{m-1} x_k^u y_k^v z_k^w a_{uvw} = 0 \qquad k = 1, \ldots, r$$

in $\mathbb{R}^{(m+2)(m+1)m/6}$ is a linear subspace of dimension $[(m+2)(m+1)m/6] - r$.

Theorem 2. *Suppose that the three-dimensional polynomial vector field X of degree m admits p invariant algebraic surfaces $f_i = 0$ with cofactors K_i for $i = 1, \ldots, p$, q exponential factors $\exp(g_j/h_j)$ with cofactors L_j for $j = 1, \ldots, q$, and r independent singular points $(x_k, y_k, z_k) \in \mathbb{R}^3$ such that $f_i(x_k, y_k, z_k) \neq 0$ for $i = 1, \ldots, p$ and for $k = 1, \ldots, r$.*

(1) If there exist $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$, then the function

$$|f_1|^{\lambda_1} \cdots |f_p|^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \cdots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} \tag{5}$$

is a first integral of the vector field X.

(2) If

$$p + q + r \geqslant \tfrac{1}{6}(m+2)(m+1)m + 1$$

then there exist $\lambda_i, \mu_j \in \mathbb{R}$ not all zero such that $\sum_{i=1}^p \lambda_i K_i + \sum_{j=1}^q \mu_j L_j = 0$.

(3) X has $[(m+2)(m+1)m/6] + 3$ invariant algebraic surfaces if and only if X has a rational first integral.

For a proof of statements (1) and (2) of this version of the Darboux theory of integrability for the three-dimensional polynomial vector fields see [14, 29]. The proofs are given in two dimensions but the arguments extend directly to higher dimensions. Statement (3) is due to Jouanolou [30] (see also Weil [31]). An improvement of statement (3) for planar polynomial vector fields can be found in [14].

To apply statement (2) of theorem 2 to a three-dimensional Lotka–Volterra system, we note that since this system always has three invariant algebraic surfaces $x = 0$, $y = 0$, $z = 0$, and $[(m+2)(m+1)m/6] + 1 = 5$, then it is sufficient for the integrability to have:

(i) two additional invariant algebraic surfaces or exponential factors;

(ii) one additional invariant algebraic surface $f_4 = 0$ such that the singular point S of the three-dimensional Lotka–Volterra system

$$\left(\frac{A\lambda - \mu - AC\nu}{1 + ABC}, \frac{-AB\lambda + B\mu - \nu}{1 + ABC}, \frac{-\lambda - BC\mu + C\nu}{1 + ABC}\right)$$

is not contained in the surfaces $x = 0$, $y = 0$, $z = 0$ and $f_4 = 0$;

(iii) one additional exponential factor when the singular point S is not contained in $x = 0$, $y = 0$ and $z = 0$.

We remark that if using statement (1) of theorem 2 we obtain invariant algebraic surfaces or exponential factors whose cofactors are linearly dependent in number smaller than five, then we also obtain a first integral of the system.

Finally, from statement (3) of theorem 2 it follows that if a three-dimensional Lotka–Volterra system has seven invariant algebraic surfaces, then it has a rational first integral.

In another context if the reader wants to better understand what kind of integrals can be obtained using the Darboux theory of integrability, see the good papers by Prelle and Singer [32] and Singer [32].

3. Invariant algebraic surfaces

In this section we study the invariant algebraic surfaces of the three-dimensional Lotka–Volterra systems of degree at most two. Thus, in proposition 3, we present the invariant planes (invariant algebraic surfaces of degree one) together with their cofactors, and the conditions for their existence. However, before we associate to a given three-dimensional Lotka–Volterra system (1) two if $ABC = 0$, or five if $ABC \neq 0$, *equivalent* three-dimensional Lotka–Volterra systems. The first two are obtained by performing a circular permutation of the variables x, y, z and of the parameters λ, μ, ν and A, B, C, and the remainding three systems are obtained by performing the transformation

$$(x, y, z, \lambda, \mu, \nu, A, B, C) \to (Bx, Az, Cy, \lambda, \nu, \mu, 1/C, 1/B, 1/A)$$

and the two new transformations are obtained from the above circular permutations. We say that all these Lotka–Volterra systems are *E equivalent*. All the results of this paper are stated modulo these *E equivalences*.

Proposition 3. *A three-dimensional Lotka–Volterra system has an invariant plane $f = 0$ with cofactor K in the following cases, modulo the E equivalences.*

(1) Any three-dimensional Lotka–Volterra system has the invariant plane $f = x = 0$ with $K = \lambda + Cy + z$.
(2) If $\lambda = \mu$, $A = 1$ and $C \neq 0$, then $f = x - Cy = 0$ and $K = z + \lambda$.
(3) For an ABC system if $ABC + 1 = 0$, then $f = ABx + y - Az = 0$ and $K = 0$; consequently f is a polynomial first integral.

Proof. The proof is obtained by finding the linear f satisfying equation (3). The second part of statement (3) follows directly from the definition of the first integral. □

The next proposition summarizes the invariant algebraic surfaces of degree two for the three-dimensional Lotka–Volterra systems, their cofactors and the conditions for their existence. In it we omit many invariant algebraic surfaces of degree two for the three-dimensional Lotka–Volterra systems that later on will not contribute to obtaining relevant information for the integrability of these systems. In fact, the relevant information for obtaining first integrals is only given by the quadratic invariant algebraic surfaces of degree two of the ABC systems, so in the following proposition we restrict our attention to such systems.

Proposition 4. *The ABC systems have the following invariant algebraic surfaces $f = 0$ of degree two with cofactor K, modulo the E equivalences.*

(1) If $B = 2$ and $A(C + 1) + 1 = 0$, then $f = 2A^2xz - (y - Az)^2 = 0$ and $K = 2x$.
(2) If $A = 1$ and $B = 2$, then $f = xz + (C + 1)y(y - z) = 0$ and $K = 2x + z$.
(3) If $ABC - 1 = 0$ and $B(A + 1) + 1 = 0$, then $f = A^2(Bx - z)^2 - 2A(Bx + z)y + y^2 = 0$ and $K = 0$; consequently f is a polynomial first integral.
(4) If $B = 1$ and $A(C + 1) + 1 = 0$, then $f = x(y + A^2Cz) - C(y - Az)(y - Az + \alpha) = 0$ and $K = x$, where α is an arbitrary constant. The system has the rational first integral $H = [x(y + A^2Cz) - C(y - Az)^2]/(y - Az)$.
(5) If $A = 1$ and $C = 1$, then $f = (x - y)(Bx - z) + \alpha x = 0$ and $K = y + z$, where α is an arbitrary constant. The system has the rational first integral $H = [(x - y)(Bx - z)]/x$.

Proof. The proof for the existence of invariant algebraic surfaces under suitable assumptions is obtained by finding the quadratic polynomials f satisfying equation (3). Moreover, under the hypotheses of statements (4) and (5), the three-dimensional Lotka–Volterra system has more than seven invariant algebraic surfaces, then from theorem 2(3) it follows that it has a rational first integral. These rational first integrals are computed using theorem 2(2). □

4. Exponential factors

Proposition 5 summarizes for the three-dimensional Lotka–Volterra system the exponential factors of the form $\exp(g/h)$, where g and h are polynomials of degree at most two, their cofactors and the conditions for their existence. In the next proposition we omit many exponential factors of the above form that later on do not contribute to obtaining relevant information for the integrability of the three-dimensional Lotka–Volterra systems.

Proposition 5. *A three-dimensional Lotka–Volterra system has the following exponential factors modulo the E equivalences.*

(1) If $\lambda + (\mu B - \nu)C = 0$ *and* $ABC + 1 = 0$, *then* $f = \exp(ABx + y - Az)$ *and* $K = (\mu B - \nu)x + \mu y - \nu Az$.
(2) If $\lambda = \mu$, $A = 1$ *and* $C = 0$, *then* $f = \exp(y/x)$ *and* $K = y$.
(3) If $\lambda = \mu + \nu$, $A = 1$ *and* $C = 1$, *then* $f = \exp[(x - y)(z - Bx)/x]$ *and* $K = -B\lambda x + B\mu y + \nu z$.
(4) If $\lambda = \mu$, $\nu = 0$, $A = 1$ *and* $C(B + 1) + 1 = 0$, *then* $f_1 = \exp[B(x - Cy) - (x + C^2By)z/(x - Cy)]$ *with* $K = B\lambda(x - Cy)$.
(5) If $\lambda = \mu$, $\nu = 0$, $A = 1$, $B = 0$ *and* $C = -1$, *then* $f = \exp[\lambda(z(\alpha x - y) + (x + y)^2)/(\lambda x + \lambda y + xz)]$ *and* $K = \lambda(x + y)$.
(6) For an ABC system with $A = -1$, $B = \frac{1}{2}$ *and* $C = 0$, *we have* $f = \exp[(y + z)^2/(xy)]$ *and* $K = y + z$.

Proof. The proof is obtained by finding the exponential factors $f = \exp(g/h)$ with g a quadratic polynomial and h one of the algebraic invariant surfaces reported in propositions 3 and 4 (see theorem 2). In order for such functions f to be exponential factors they must satisfy equation (4) with a cofactor given by a polynomial of degree at most one. So, solving equation (4) the proposition follows. □

5. First integrals and integrable systems

In this section we apply the Darboux theory of integrability described in section 2 to the three-dimensional Lotka–Volterra systems. Theorem 6 exhibits for these systems the first integrals obtained by using only invariant algebraic surfaces of degrees one or two (given by propositions 3 and 4, respectively) and statement (1) of theorem 2. More precisely, we look for first integrals of the form

$$|f_1|^{\lambda_1}|f_2|^{\lambda_2}|f_3|^{\lambda_3}|f_4|^{\lambda_4} \tag{6}$$

where $f_4 = 0$ is an invariant algebraic surface of degree at most two, and the λ_i are real numbers not all zero.

Theorem 6. *A three-dimensional Lotka–Volterra system has the following first integrals of the form (6) where $f_i = 0$ are invariant algebraic surfaces of degree at most two modulo the E equivalences.*

(1) *If $\lambda + (\mu B - \nu)C = 0$ and $ABC + 1 = 0$, then $H = |x|^A y^{-1} |z|^{-AC}$.*
(2) *If $\lambda = \mu$, $\nu = 0$, $A = 1$ and $C \neq 0$, then $H = x|y|^{BC}|z|^{-C}|x - Cy|^{-BC-1}$.*
The next four statements hold for ABC systems.
(3) *If $ABC + 1 = 0$, then it is integrable with the first integrals $H_1 = ABx + y - Az$ and $H_2 = xy^{BC}z^{-C}$.*
(4) *If $ABC - 1 = 0$ and $B(A + 1) + 1 = 0$, then $H = A^2(Bx - z)^2 - 2A(Bx + z)y + y^2$.*
(5) *If $B = 2$ and $A(C + 1) + 1 = 0$, then $H = x^2|y|^{2(C+1)}|z|^{-2C}|2A^2xz - (y - Az)^2|^{C-1}$.*
(6) *If $A = 1$ and $B = 2$, then it is integrable with the first integrals $H_1 = x|y|^{-2(C+1)}|z|^{-C} |xz + (C + 1)y(y - z)|^{2C+1}$ and $H_2 = (x - Cy)[xz + (C + 1)y(y - z)]/y^2$.*

Proof. Using theorem 2(1) we construct the first integrals of the form (6) for each statement of theorem 6, where $f_i = 0$ is an invariant algebraic surface with cofactor K_i given by propositions 3 or 4, and the following equality holds $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 + \lambda_4 K_4 = 0$ for some $\lambda_i \neq 0$. Of course, for the statements that two independent first integrals exist the corresponding three-dimensional Lotka–Volterra system becomes integrable. For statement (3) H_1 is given directly in proposition 3 (3) and H_2 can be found, for instance, in [26, 27, 34]. □

Theorem 7 exhibits for the three-dimensional Lotka–Volterra system first integrals of the form

$$|f_1|^{\lambda_1}|f_2|^{\lambda_2}|f_3|^{\lambda_3} \exp\left(\frac{g}{h}\right) \tag{7}$$

where $\exp(g/h)$ is an exponential factor given by proposition 5, and the $f_i = 0$ are invariant algebraic surfaces of degree at most two given by propositions 3 and 4.

Theorem 7. *A three-dimensional Lotka–Volterra system has the following first integrals of the form (7) where f_i, g and h are polynomials of degree at most two, modulo the E equivalences.*

(1) *If $\lambda + (\mu B - \nu)C = 0$ and $ABC + 1 = 0$, then it is integrable with the first integrals given by theorem 6(1) and $H = |y|^\nu |z|^{-\mu} \exp(ABx + y - Az)$.*
(2) *If $\lambda = \mu$, $A = 1$ and $BC = -1$, then $H = |y + Bx|^\nu |z|^{-\lambda} \exp(Bx + y - z)$.*
(3) *If $\lambda = \mu$, $\nu = 0$, $A = 1$ and $C = 0$, then $H = |x|^{-B}|y|^B z^{-1} \exp(y/x)$.*
(4) *If $\lambda = \mu$, $\nu = 0$, $A = 1$ and $B = 1$, then $H = |y|^\lambda |x|^{-\lambda} \exp[(y - z)(Cy - x)/y]$. Moreover, if $C \neq 0$ or $C = 0$, then it is integrable with the additional first integral given by theorem 6(2) or in (3), respectively.*
(5) *If $\lambda = \mu$, $\nu = 0$, $A = 1$ and $C = 1$, then if $B \neq 0$ it is integrable with the first integrals given by theorem 6(2) and $H = |x|^{\mu B}|y|^{-\mu B} \exp[(x - y)(Bx - z)/x]$.*
(6) *If $\lambda = \mu$, $\nu = 0$, $A = 1$ and $BC = -1$, then it is integrable with the first integrals given by theorem 6(2) and (2).*
(7) *If $\lambda = \mu$, $\nu = 0$, $A = 1$ and $C(B + 1) + 1 = 0$, then it is integrable with the first integrals given by theorem 6(2) and $H = |x|^{\mu B}|y|^{-\mu B} \exp[B(x - Cy) - (x + C^2By)z/(x - Cy)]$.*
(8) *If $\lambda = 2\mu$, $\nu = \mu$ and $A = B = C = 1$, then $H = |x|^\mu |y - z|^{-2\mu} \exp[(x - z)(x - y)/x]$.*
(9) *If $A = -1$, $B = \frac{1}{2}$ and $C = 0$ then the ABC systems have the first integral $H = xy^{-1}z^2 \exp[-2(y + z)^2/(xy)]$.*

Proof. We note that, by construction, at least one of the first integrals of each statement of theorem 7 is formed by three invariant algebraic surfaces $f_i = 0$ with cofactors K_i $(i = 1, 2, 3)$, given by propositions 3 and 4, and one exponential factor $f_4 = \exp(g/h)$ with cofactor K_4 given by proposition 5. For each statement, a solution of system $\lambda_1 K_1 + \lambda_2 K_2 + \lambda_3 K_3 + K_4 = 0$ exists. Hence, by theorem 2(1), to each statement corresponds a first integral of the form $|f_1|^{\lambda_1}|f_2|^{\lambda_2}|f_3|^{\lambda_3} f_4$. Of course, for the statements which have two independent first integrals of the form given by the expressions (6) or (7), the corresponding three-dimensional Lotka–Volterra system becomes integrable. □

In the case $\lambda = \mu = \nu = 0$ an easier expression for the first integral in statements (5) and (7) of theorem 7 can be found in statements (5) and (4) of proposition 4, respectively.

6. About the known first integrals and integrable systems

Now we study which first integrals of the three-dimensional Lotka–Volterra systems and which integrable three-dimensional Lotka–Volterra systems given in propositions and theorems 3–7, are new.

Here we call *Darboux-type functions* the functions of the form (5), where the f_i's, g_i's and h_i's are polynomials in the variables x, y and z.

Grammaticos *et al* [24] classify in 20 classes the three-dimensional Lotka–Volterra systems for which they find first integrals or integrability. They found these first integrals using three different methods: the linear compatibility method, Painlevé analysis and the Jacobi last-multiplier method. In what follows we compare their results with ours. Since the systems having $\lambda = \mu = \nu \neq 0$ can be studied through the same system taking $\lambda = \mu = \nu = 0$ as was mentioned in the introduction, we identify both systems. Then we have:

- For systems 1 [24] they give a first integral and we prove that such systems are integrable, see theorem 7(1) taking $\lambda = \mu = \nu = 0$.
- The integrable systems 2 and 3 [24] are contained in the systems of theorem 7(4) taking $\lambda = \mu = \nu = 0$.
- The integrable systems 4 [24] are the systems of theorem 7(1).
- The first integral of systems 5 [24] with $\lambda = \mu$ is the first integral of theorem 7(2). The integrable systems 5 [24] with $\lambda = \mu = \nu$ are contained in the systems of theorem 7(1) taking $\lambda = \mu = \nu = 0$. The integrable systems 5 [24] with $\nu = 0$ are the systems of theorem 7(6). The first integral of systems 5 [24] with $\lambda = \mu = 0$ cannot be obtained with the Darboux theory of integrability described in theorem 2, because such a first integral is not a Darboux-type function.
- The integrable systems 6 and 7 [24] are contained in the system (6) of theorem 6, modulo E equivalences.
- The first integrals of the integrable systems 8 [24] cannot be obtained with the techniques of this paper because they are not Darboux-type functions. For these systems we have the first integral of proposition 4(3).
- One of the first integrals of the integrable systems 9 [24] and 10 [24] is obtained in theorems 6(2) and 7(3), respectively; the other is not of Darboux-type.
- One of the first integrals of the integrable systems 11 [24] is obtained in theorem 6(2) when $C \neq 0$ and in theorem 7(3) when $C = 0$, respectively; the other is not of Darboux-type.
- The first integral of systems 12 [24] is the first integral of theorem 7(8) up to circular permutation.
- The integrable systems 13 [24] are the systems of theorem 7(5) up to circular permutation.

- The integrable systems 14 [24] is the system of theorem 7(4) and 15 [24] is contained in theorem 7(7), respectively.
- The first integrals of systems 16–20 [24] can be obtained with the Darboux theory of integrability, but since they are polynomials of degree greater than two we have not studied them in this paper.

Almeida *et al* [23] classify in 11 classes the three-dimensional Lotka–Volterra systems for which they find time-dependent and time-independent first integrals. They found these first integrals using the method of Lie symmetries. We remark that all the time-dependent first integrals that they provided correspond to systems having $\lambda = \mu = \nu \neq 0$. When we identify these systems with the same systems but having $\lambda = \mu = \nu = 0$ through the change of variables (2), all the time-dependent first integrals become time independent. In this paper they are identified. Then we obtain for the results which appear in their tables II and IV.

- The possible first integrals of systems II(1) [23] cannot be exhibited, neither in 22 nor with the techniques of this paper.
- Systems II(2)–(5) [23] with $\lambda = \mu = \nu = 0$, for which they give a first integral, we prove that they are integrable, see theorem 7(1).
- For systems II(6) and II(8) [23] with $\lambda = \mu = \nu = 0$, we prove that they are integrable, see theorem 7(4).
- For systems II(7) [23] with $\lambda = \mu = \nu = 0$, we exhibit a first integral, see theorem 6(4).
- For systems II(9) [23] if $\nu = 0$ and $A = 1$, we prove that they are integrable, see theorem 7(6).
- The first integral of systems IV(1) and IV(5) [23] is obtained in theorem 6(2) taking $\lambda = \mu = \nu = 0$.
- The integrable systems IV(2) and IV(6) [23] are contained in theorem 7(4) taking $\lambda = \mu = \nu = 0$.
- The integrable systems IV(3) and IV(7) [23] are contained in theorem 6(6) taking $\lambda = \mu = \nu = 0$.
- The first integrals of systems IV(4) and IV(11) [23] are contained and coincide with the integrable systems of theorem 7(7), respectively.
- The integrable system IV(8) [23] is contained in theorem 6(6) modulo E equivalences, taking $\lambda = \mu = \nu = 0$.
- The first integral of systems IV(9) [23] is obtained in theorem 6(2).
- The integrable systems IV(10) [23] is contained in theorem 7(4).

Labrunie [27] and Moulin-Ollagnier [28] characterize all the polynomial first integrals of the ABC systems. For degrees one and two they are the first integrals of theorems 6(3) and 6(4), respectively.

In the papers of Cairó *et al* [25] and Cairó and Feix [26] the authors studied first integrals of the n-dimensional Lotka–Volterra systems containing all the quadratic self-interacting terms. These first integrals in the particular case of dimension three and when the quadratic self-interacting terms vanish, coinciding with the first integral of proposition 3(3).

7. Conclusion

We have presented in section 2 a summary of how to use the Darboux theory of integrability for studying the first integrals of three-dimensional polynomial differential systems. As was mentioned this theory can be applied to n-dimensional polynomial differential systems. We apply this theory only up to degree two for the invariant algebraic surfaces and the exponential factors. This is enough to obtain the major part of the first integrals found until now for our three-dimensional Lotka–Volterra system, and to detect new cases of integrability. Some of these cases were actually already known as cases of partial integrability only (see theorems 6 and 7, and section 6). In particular, we find first integrals for the following three new cases for the *ABC* systems: theorems 6(5) and 6(6) (which generalize the cases 6 and 7 of Grammaticos *et al* and the cases IV(7) and IV(8) of Almeida *et al*) and theorem 7(9).

Acknowledgment

JL is partially supported by a DGYCIT grant number PB96-1153.

References

[1] Ziglin S L 1983 Branching of solutions and nonexistence of first integrals in Hamiltonian mechanics I *Func. Anal. Appl.* **16** 181
Ziglin S L 1983 Branching of solutions and nonexistence of first integrals in Hamiltonian mechanics II *Func. Anal. Appl.* **17** 6
[2] Yoshida H 1987 A criterion for the nonexistence of an additional integral in Hamiltonian systems with a homogeneous potential *Physica* D **29**128
[3] Cantrijn F and Sarlet W 1981 Generalizations of Noether's theorem in classical mechanics *SIAM Rev.* **23** 467
[4] Darboux G 1878 Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré Mélanges *Bull. Sci. Math.* series 2 **2** 60, 123, 151
[5] Olver P J 1986 *Applications of Lie Groups to Differential Equations* (New York: Springer)
[6] Bountis T, Grammaticos B and Ramani A 1989 On the complete and partial integrability of non-Hamiltonian systems *Phys. Rep.* **180** 159
[7] Lax P D 1968 Integrals of nonlinear equations of evolution and solitary waves *Commun. Pure Appl. Math.* **21** 467
[8] Hietarinta J 1987 Direct methods for the search of the second invariant *Phys. Rep.* **147** 8
[9] Strelcyn J M and Wojciechowski S 1988 A method of finding integrals for three-dimensional dynamical systems *Phys. Lett.* A **133** 207
[10] Carleman T 1932 Application de la théorie des équations intégrales linéaires aux systémes d'équations différentielles non linéaires *Acta Math.* **59** 63
[11] Andrade R F S and Rauh A 1981 The Lorenz model and the method of Carleman embedding *Phys. Lett.* A **82** 276
[12] Brenig L and Goriely A 1989 Universal canonical forms for time-continuous dynamical systems *Phys. Rev.* A **40** 4119
[13] Christopher C J 1994 Invariant algebraic curves and conditions for a center *Proc. R. Soc. Edin.* A **124** 1209
[14] Christopher C J and Llibre J 1999 Algebraic aspects of integrability for polynomial systems *Qualit. Theory Dynam. Syst.* **1** 71–95
[15] Chavarriga J, Llibre J and Sotomayor J 1997 Algebraic solutions for polynomial vector fields with emphasis in the quadratic case *Expositiones Math.* **15** 161
[16] Man Yu 1993 Computing closed-form solutions of first-order ODEs using the Prelle–Singer procedure *J. Symb. Comput.* **16** 423
[17] Schlomiuk D 1993 Elementary first integrals and algebraic invariant curves of differential equations *Expositiones Math.* **11** 433
[18] Cairó L and Llibre J 1999 *Integrability and Algebraic Solutions for the 2-D Lotka–Volterra System (Monographs in Physics vol 518)* (Berlin: Springer) p 243
[19] Cairó L, Feix M R and Llibre J 1999 Darboux method and search of invariants for the Lotka-Volterra and complex quadratic systems *J. Math. Phys.* **40** 2074–91
[20] Cairó L, Feix M R and Llibre J 1999 Integrability and algebraic solutions for planar polynomial differential systems with emphasis on the quadratic systems *IME-U5P Resenhas* **4** 127–61
[21] Volterra V 1931 *Leçons sur la Théorie Mathématique de la Lutte pour la Vie* (Paris: Gauthier–Villars)
[22] Lotka A J 1920 Analytical note on certain rhythmic relations in organic systems *Proc. Natl Acad. Sci.* **6** 410
[23] Almeida M A, Magalhães M E and Moreira I C 1995 Lie symmetries and invariants of the Lotka–Volterra system *J. Math. Phys.* **36** 1854

[24] Grammaticos B, Moulin-Ollagnier J, Ramani A, Strelcyn J M and Wojciechowski S 1990 Integrals of quadratic ordinary differential equations in R^3: the Lotka–Volterra system *Physica* A **163** 683

[25] Cairó L, Feix M R and Goedert J 1989 Invariants for models of interacting populations *Phys. Lett.* A **140** 421

[26] Cairó L and Feix M R 1992 Families of invariants of the motion for the Lotka–Volterra equations: the linear polynomials family *J. Math. Phys.* **33** 2440

[27] Labrunie S 1996 On the polynomial first integrals of the (a, b, c) Lotka–Volterra system *J. Math. Phys.* **37** 5539

[28] Moulin-Ollagnier J 1997 Polynomial first integrals of the Lotka–Volterra system *Bull. Sci. Math.* **121** 463

[29] Llibre J and Rodríguez G 2000 Darboux integrability and invariant hyperplanes for d-dimensional polynomial differential systems *Bull. Sci. Math.* at press

[30] Jouanolou J P 1979 *Equations de Pfaff Algébriques (Lectures Notes in Mathematics vol 708)* (Berlin: Springer)

[31] Weil J A 1995 Constantes et polynômes de Darboux en algèbre différentielle: applications aux systèmes différentiels linéaires *PhD École Polythecnique*

[32] Prelle M J and Singer M F 1983 Elementary first integrals of differential equations *Trans. Am. Math. Soc.* **279** 215

[33] Singer M F 1992 Liouvillian first integrals of differential equations *Trans. Am. Math. Soc.* **333** 673

[34] Moulin-Ollagnier J 1999 Rational integration of the Lotka–Volterra system *Bull. Sci. Math.* **123** 437